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On the enhancement of centrifugal separation

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We consider the two-phase flow of a suspension in a rotating cylinder with inclined endplates for which inertial and viscous effects are small. It is shown that, when the Coriolis force is dominant, flow in the core is essentially unaffected by geometry. If a fluid particle can make a complete circuit about the rotation axis, the sedimentation velocity cannot be augmented by geometrical effects as it can in gravitational settling. However, with the insertion of a complete meridional barrier to block movement around the centre, separation becomes more sensitive to the shape of the container walls. In this case, behaviour similar to that in a gravitational field is possible once again.

1. Introduction

Centrifugal separation of a two-phase fluid is an important process with many industrial applications. After a century of empirical developments, modern centrifuges work very well, and this is a credit to engineering practices because neither the separation nor the flow of a two-phase fluid is really understood in satisfactory detail. Design improvements are still possible, and for this reason a better understanding of the process is of great significance.

In recent investigations of separation based on two-phase flow theory (Greenspan 1983; Ungarish & Greenspan 1984) the motion of a mixture in a long axisymmetric rotating cylinder was considered. In the present work we examine the potential for enhancing separation by geometrical modifications of the container, namely the inclination of the endcaps and a meridional barrier.

Improved separative performance is achieved in gravitational settlers by inclining the walls of the container with respect to the direction of the gravitational field. This is known as the 'Boycott effect' (most recently analysed by Acrivos & Herbolzheimer 1979, where other important references are cited). Enhanced sedimentation, driven by the formation of a pure-fluid layer near the downward-inclined wall as shown in figure 1, can be regarded as a result of the increase in the surface area available for creating particle-free fluid. (This is the interpretation most easily generalized to describe centrifugal separation.) In circumstances typical of many problems, the instantaneous settling rate turns out to be proportional to the horizontal projection of the interfacial area between the pure fluid and the mixture. However, this interesting, purely kinematic conclusion is of little use without a dynamic understanding of the flow field that allows determination of the interface. The detailed analysis of Probstein, Yung & Hicks (1977) shows that the pure (lighter) fluid zone is a thin boundary layer on the downward inclined wall. The purified fluid in this layer is set in a fast upward convective motion by the hydrostatic pressure in the adjacent heavier mixture. Volume conservation requires a corresponding downward flux in the

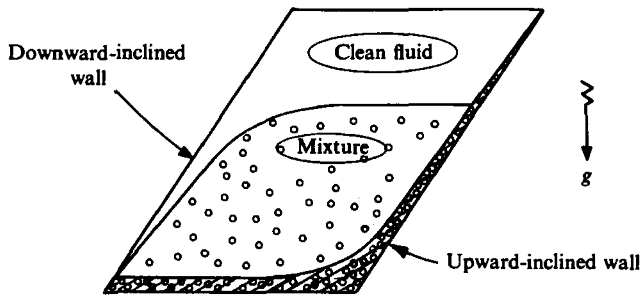

FIGURE 1. Gravity-settling in a container with inclined walls.

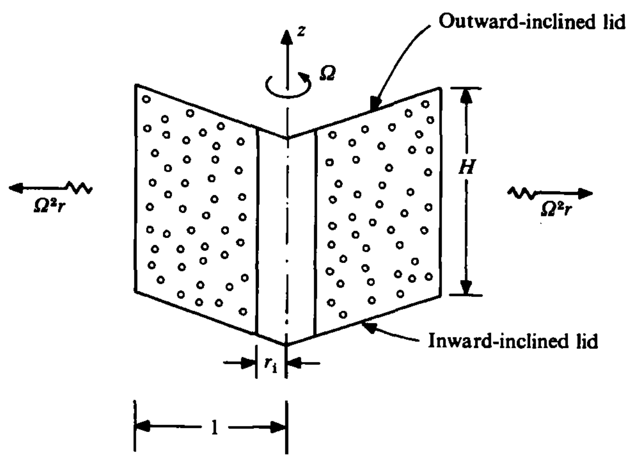

FIGURE 2. Centrifugal separation in a cylindrical container with inclined lids.

mixture bulk, which, in turn, carries the dispersed particles with a velocity that is considerably larger than the conventional terminal settling velocity in the quiescent fluid.

One might expect the flow in a rotating container whose lids are inclined with respect to the direction of the centrifugal force (figure 2) to be similar to that in the abovementioned example of gravitational settling. However, to our knowledge, no convective augmentation of settling in a rotating container has been observed, except for flows in *very* narrow gaps, where viscous effects prevail throughout as discussed by Bark & Johansson (1982). There are basic reasons for this interesting dissimilarity.

In the typical systems considered by Probstein *et al.* (1977), Hill, Rothfus & Kun (1977), Acrivos & Herbolzheimer (1979) and Schneider (1982), the momentum balance in the mixture bulk is mainly between buoyancy and pressure, while inertial and viscous forces are important only in thin boundary layers. As a consequence, the flow of the mixture is primarily controlled by the requirements of kinematic continuity and compatibility with the boundary conditions. This provides a strong connection between the geometry of the boundaries and the basic flow that underlies the enhanced gravitational settling that we call the Boycott effect. Moreover, the volume fraction of the mixture remains constant throughout the duration of the settling process, and the geometry essentially affects only the velocity of the mixture.

The flow observed in the cylindrical rotating system considered here is different, even though the inertial and viscous forces in the mixture domain are also small in comparison with that due to buoyancy. First of all, the volume fraction α cannot be a constant in a centrifugal force field. More importantly, the Coriolis acceleration strongly couples the dynamics with the kinematics. For example, the radial mass flux in an axisymmetric cylinder is severely restricted because the pressure gradient can have no azimuthal component. Flow between source and sink is then via non-divergent Ekman layers at the endwalls which are supported by an interior motion that is, essentially, an inviscid vortex. However, the insertion of a complete meridional barrier dramatically alters the motion. In particular, an azimuthal pressure gradient is now available to counteract the corresponding non-zero component of the Coriolis acceleration, which is proportional to the radial velocity. The barrier prevents azimuthal motion and produces a radial momentum balance once again between buoyancy and pressure terms, in close resemblance to the situation for gravitational settling. The flow of the mixture in a sectioned cylinder can then exhibit a considerable convective augmentation of settling due to an inclination of the endwalls.

2. Equations of motion

Consider the time-dependent motion of a mixture of two incompressible constituents. The dispersed phase consists of small particles (or droplets) of approximately constant radius a^* and occupies the volume fraction α. The averaged flow variables of the continuous and dispersed phase are denoted by subscripts C and D, while a variable of the mixture is unsubscripted (e.g. the densities ρ_D^*, ρ_C^* and ρ^*, where $\rho^* = (1-\alpha)\rho_C^* + \alpha\rho_D^*$; an asterisk designates a dimensional variable).

Let $\boldsymbol{q}^* = u^*\hat{\boldsymbol{r}} + v^*\hat{\boldsymbol{\theta}} + w^*\hat{\boldsymbol{z}}$ be the fluid velocity in a cylindrical coordinate system rotating with constant angular velocity Ω^* around the z-axis and let $\boldsymbol{j}^* = (j_r^*, j_\theta^*, j_z^*)$ be the corresponding volume flux. (Some useful kinematic relations for velocities and fluxes are given in the Appendix.)

The primary variable of the process under consideration is the relative velocity $\boldsymbol{q}_R^* (= \boldsymbol{q}_D^* - \boldsymbol{q}_C^*)$. A reasonable formula for $\boldsymbol{q}_R^*$ is based on the drag–buoyancy balance and the assumptions that the local drag on a small particle is approximated by Stokes' law and that the effective viscosity of the fluid depends only on the local volume fraction α. Therefore

$$\boldsymbol{q}_R^* = \epsilon\beta\Omega^* r^* f(\alpha)\,\hat{\boldsymbol{r}}, \tag{2.1}$$

where

$$\epsilon = \frac{\rho_D^* - \rho_C^*}{\rho_C^*}, \quad \beta = \frac{2}{9}\frac{a^{*2}}{\nu^*/\Omega^*}, \tag{2.2}$$

ν^* is the kinematic viscosity of the fluid and $\lim_{\alpha\to 0} f(\alpha) = 1$.

The dimensionless parameter β, a modified Taylor number, measures the ratio of the particle size to the thickness of the Ekman layer, or equivalently the ratio of the Coriolis force and Stokes drag on a particle. Rotational effects may strongly affect the drag on a particle for β moderate or large; *the following analysis is restricted to small values of* β. (We note in passing that the investigations of Herron, Davis & Bretherton (1975) and Karafilian & Kotas (1981) have shown the validity of (2.1) in the limit of one particle, for small β.) The function $f(\alpha)$ accounts for the small-scale interaction between the fluid and particles, and (according to Ishii & Chawla 1979),

$$f(\alpha) = (1-\alpha)\left(1 - \frac{\alpha}{\alpha_M}\right)^{2.5\alpha_M}, \tag{2.3}$$

where α_M is the maximal-packing volume fraction. This particular choice is not essential in what follows.

Introduce dimensionless variables obtained by scaling the velocities by $|\epsilon|\beta\Omega^* r_0^*$, the typical value of $|\boldsymbol{q}_R^*|$; lengths by r_0^*, the outer radius of the cylindrical container; time by $(|\epsilon|\beta\Omega^*)^{-1}$ and density by ρ_C^*. The equations of continuity are then

$$\frac{\partial\alpha}{\partial t}+\nabla\cdot\boldsymbol{j}_D = 0, \tag{2.4}$$

$$\nabla\cdot\boldsymbol{j} = 0. \tag{2.5}$$

The rotational acceleration terms are readily incorporated in the averaged equations of motion as developed, for instance, by Ishii (1975). If, for definiteness, the stress term is assumed to be that of a constant-viscosity Newtonian fluid, then the dimensionless momentum equation for the mixture is

$$(1+\epsilon\alpha)\left\{2\hat{\boldsymbol{z}}\times\boldsymbol{q}+|\epsilon|\beta\left[\frac{\partial \boldsymbol{q}}{\partial t}+\tfrac{1}{2}\nabla(\boldsymbol{q}\cdot\boldsymbol{q})+(\nabla\times\boldsymbol{q})\times\boldsymbol{q}\right]\right\}$$
$$= -\frac{1}{\beta}\nabla P+\frac{s}{\beta}\alpha r\hat{\boldsymbol{r}}+E[\tfrac{4}{3}\nabla(\nabla\cdot\boldsymbol{q})-\nabla\times(\nabla\times\boldsymbol{q})]-|\epsilon|\beta\nabla\cdot(\alpha(1-\alpha)\boldsymbol{q}_R\boldsymbol{q}_R), \tag{2.6}$$

where $$E=\frac{\mu^*}{\rho_C^*\Omega^* r_0^{*2}},\quad P=\frac{1}{(\Omega^* r_0^*)^2|\epsilon|}\left(\frac{P^*}{\rho_C^*}-\tfrac{1}{2}\Omega^{*2}r^{*2}\right),\quad s=\frac{\epsilon}{|\epsilon|}, \tag{2.7}$$

$\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{r}}$ are unit vectors and μ^* is the effective viscosity coefficient of the mixture. The last term in (2.6) arises from the diffusion of momentum due to the relative motion; here (2.8) can be used to replace $\boldsymbol{q}_R\cdot\boldsymbol{q}_R$ in the square brackets. The Ekman number E, which measures the relative importance of viscous forces compared with the Coriolis acceleration, is assumed small. The volume fraction α is regarded as an $O(1)$ variable, and the value of $|\epsilon|$ will be considered later. It has been assumed that β is small, but it is important to keep in mind that, although $|\boldsymbol{q}_R\cdot\hat{\boldsymbol{r}}|$ is $O(1)$, the order of magnitude of the scaled velocity components remains to be calculated. In any case, the momentum-diffusion term in (2.6) can be ignored because its only (radial) component is much smaller than the buoyancy term, even for large $|\epsilon|$.

Using the dimensionless form of (2.1),

$$\boldsymbol{q}_R = sf(\alpha)\,r\hat{\boldsymbol{r}}, \tag{2.8}$$

and the kinematic relations listed in the Appendix, we find that

$$\boldsymbol{q}_D = \boldsymbol{q}+s\frac{1-\alpha}{1+\epsilon\alpha}f(\alpha)\,r\hat{\boldsymbol{r}}, \tag{2.9}$$

$$\boldsymbol{j}_D = \alpha\boldsymbol{j}+s\phi(\alpha)\,r\hat{\boldsymbol{r}}, \tag{2.10}$$

$$\boldsymbol{j} = \boldsymbol{q}-|\epsilon|\frac{\phi(\alpha)}{1+\epsilon\alpha}r\hat{\boldsymbol{r}}, \tag{2.11}$$

where $$\phi(\alpha) = \alpha(1-\alpha)f(\alpha).$$

The substitution of (2.10) in (2.4) and (2.5) yields the simplified form of the equation for the volume fraction

$$\frac{\partial\alpha}{\partial t}+\boldsymbol{j}\cdot\nabla\alpha+s\frac{1}{r}\frac{\partial}{\partial r}[\phi(\alpha)\,r^2] = 0. \tag{2.12}$$

Further reduction of the momentum equation is necessary for progress. To this end, the inertial terms are assumed negligibly small. This imposes certain restrictions on ϵ and β which will be discussed shortly. The solution of the remaining system, which is still a formidable task, is facilitated by boundary-layer theory in which the flow field is separated into inviscid and viscous domains.

Motion in the boundary layer is extremely complicated because in many cases of interest the nonlinear terms are important. Even the appropriate forms of the averaged equations and boundary conditions for two-phase rotating boundary layers are uncertain in some circumstances (Ungarish & Greenspan 1984). However, some insight can be gained by making reasonable assumptions about the boundary-layer flow.

Motion in the inviscid interior is of principal interest, and there, if inertial terms are truly small, the momentum equation (2.6) reduces to a balance between Coriolis, pressure and buoyancy terms:

$$2(1+\epsilon\alpha)\,\hat{\boldsymbol{z}}\times\boldsymbol{q} = -\frac{1}{\beta}\nabla P+\frac{s}{\beta}\,r\alpha\hat{\boldsymbol{r}}. \tag{2.13}$$

The main task is to solve for the mixture flow field as described by the system (2.11)–(2.13). The motion of the dispersed phase can then be obtained from (2.9) or (2.10). The boundary conditions on the flow will be discussed later. As initial conditions, it is assumed for simplicity that at $t = 0$ the mixture occupies the whole container, with a constant volume fraction α_{I} and is in a state of rigid rotation, i.e. $\boldsymbol{q} = 0$.

Preliminary to a detailed analysis, we note that $\alpha = \text{constant} \neq 0$ does not satisfy (2.12), which does, however, have in the limit $\alpha \to 0$ the particular space-independent solution, $\alpha = \alpha_{\mathrm{I}} \exp(-2st)$. This illustrates an important feature of the centrifugal process, namely that significant separation occurs in an $O(1)$ time period when the heavier component is squeezed out of the mixture. In addition to the 'squeezing' mechanism, separation takes place via complete disengagement, that is, in the formation of a purified region of the continuous phase that is separated from the mixture bulk by a surface of discontinuity or kinematic shock. Only the latter mechanism is significant in typical gravitational settling.

Since centrifugal separation is intrinsically a function of the radial motion, enhancement of the disengagement rate requires u_{D} or $\hat{\boldsymbol{r}}\cdot\boldsymbol{j}_{\mathrm{D}}$ to be large. In view of (2.9) and (2.10), this means that the radial velocity and volume flux of the mixture should be at least comparable to the sedimentation speed. Consequently, special attention must be paid to u (or j_r) in the subsequent analysis. In this respect flows in cylinders with and without meridional barriers (figure 3) are different and these configurations are considered separately.

3. Axisymmetric containers

Consider the linear 'inviscid' flow within the axisymmetric container shown in figure 3(*a*), for which the momentum equations are supposedly

$$-2(1+\epsilon\alpha)\,v = -\frac{1}{\beta}\frac{\partial P}{\partial r}+\frac{s}{\beta}\,r\alpha, \tag{3.1}$$

$$2(1+\epsilon\alpha)\,u = 0, \tag{3.2}$$

$$0 = -\frac{\partial P}{\partial z}. \tag{3.3}$$

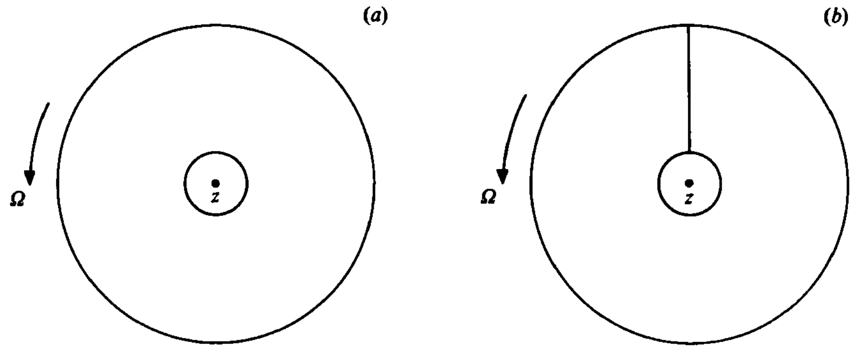

FIGURE 3. Cross-section of the cylindrical container: (a) axisymmetric configuration; (b) with complete meridional barrier.

Equation (3.1) indicates that a rapid azimuthal motion develops. During the separation process the heavier phase concentrates near the outer periphery, and, because the relative azimuthal motion between the phases is negligible (see (2.1)), conservation of angular momentum requires a backward rotation in the mixture with $v = O(1/\beta)$. However, for an azimuthal velocity of this magnitude, the inertial terms in the momentum equation can be neglected only if $|\epsilon| \ll 1$. Therefore a formal linearization procedure for symmetric configurations requires expansions in powers of ϵ, in which case the lowest-order theory would be (3.1)–(3.3) with $\epsilon \equiv 0$.

Equation (3.2) and an order-of-magnitude analysis of the inertial terms in the azimuthal equation show that in fact $u = O(|\epsilon|)$. According to (2.9), the radial velocity of the dispersed phase is then nearly equal to the slip (or drift) velocity, $u_D \approx s(1-\alpha)f(\alpha)\,r$.

Since the radial velocity of the mixture is $O(|\epsilon|)$ in an axisymmetric container, the settling velocity cannot be increased by an interior circulation. This conclusion is made irrespective of the boundary conditions, i.e. the inclination of the lids has little influence on the flow or particle-settling velocity in the inviscid core when the Coriolis force is important. An equivalent statement is that the pure-fluid layer adjacent to the inclined walls away from which particles are driven, does not remain thin, in contrast with the corresponding region near the downward-inclined wall in gravitational settling (figure 1).

It is noteworthy that addition of the linearized momentum equations yields

$$\frac{\partial}{\partial z}\left[2v+\frac{s}{\beta}\,r\alpha\right]=0, \tag{3.4}$$

which is similar to the thermal-wind equation in stratified rotating fluids.

The continuity equation implies that u and w are equally and negligibly small in the interior, which allows the reduction of (2.12) to

$$\frac{\partial \alpha}{\partial t}+sr\phi'(\alpha)\,\frac{\partial \alpha}{\partial r}=-2\phi(\alpha)\,s. \tag{3.5}$$

The solution of this equation, obtained by the method of characteristics, is

$$t=-\frac{s}{2}\int_{\alpha_\gamma}^{\alpha}\frac{\mathrm{d}\alpha}{\phi(\alpha)}+\gamma, \tag{3.6}$$

along $$r = \frac{r_\gamma(\gamma)}{[\phi(\alpha(t))]^{\frac{1}{2}}}[\phi(\alpha_\gamma)]^{\frac{1}{2}}, \quad z = z_\gamma, \tag{3.7}$$

where, at time $t = \gamma$, $\alpha = \alpha_\gamma$ at $r = r_\gamma$, $z = z_\gamma$.

In the particular case when $\alpha = \alpha_{\mathrm{I}}$ is a constant it follows that subsequently $\alpha = \alpha(t)$ in the separating region. This corresponds in the limit $|\epsilon| \to 0$ to the similarity solution given by Greenspan (1983). However, in that work the radial volume flux density j_r is identically zero as a consequence of the similarity assumption, whereas in the present case the radial momentum balance only constrains this quantity to be small, $O(|\epsilon|)$.

Ekman boundary layers on the inclined walls are required to adjust the large azimuthal velocity of the core to the boundary conditions. This complex interaction, which will be described elsewhere, depends crucially on the value of $\lambda = (E^{\frac{1}{2}}/H)/|\epsilon|\beta$, i.e. the ratio of the separation and spin-up times (H is the characteristic height of the container). When λ is not large the major conclusions stated above are unaffected by the secondary boundary-layer flow.

The analysis becomes quite complicated when the condition of axial symmetry is relaxed but the container allows a particle to make a complete circuit about the rotation axis. Some progress can be made by considering azimuthal averages of the flow-field variables, defined by

$$\langle \psi \rangle = \frac{1}{2\pi}\int_0^{2\pi} \psi \,\mathrm{d}\theta. \tag{3.8}$$

Averaging eliminates the terms that contain derivatives of θ, and we deduce from (2.6) that $\langle u \rangle = O(\epsilon)$. Consequently the averaged 'Boycott effect' must also be negligibly small.

This analysis, based as it is on the assumption $|\epsilon| \ll 1$, does not completely reject the possibility of a convective enhancement of separation for larger values of the density parameter. On the other hand, this is unlikely because the *retrograde* azimuthal velocity of the mixture increases with $|\epsilon|$, thereby reducing considerably the effective centrifugal 'gravity' $[\Omega^*(1+|\epsilon|\beta v)]^2$. Improved separation can be obtained in the parameter range $|\epsilon| = O(1)$ by obstructing the azimuthal motion of the mixture with a meridional barrier. This configuration is discussed next.

4. Meridionally sectioned containers

Consider a cylindrical container divided by a complete meridional barrier at $\theta = 0$, as shown in figure 3(*b*). (The extension to several barriers is straightforward.) Since the azimuthal motion of the mixture is obstructed, it is anticipated that the radial velocity component is $O(1)$, with $v = O(1)$ at most. Hence the nonlinear terms in the momentum equations are $O(|\epsilon|\beta)$, and because β is small the restriction required on ϵ in §3 can be relaxed to $|\epsilon| = O(1)$. This suggests the expansion

$$f(\boldsymbol{r}, t) = f_0(\boldsymbol{r}, t) + \beta f_1(\boldsymbol{r}, t) + \dots$$

for each of the dependent variables.

The substitution of the expansions in (2.5), (2.12) and (2.13) with $E = 0$ yields

$$0 = -\frac{\partial P_0}{\partial r} + sr\alpha_0, \quad 0 = \frac{\partial P_0}{\partial \theta} = \frac{\partial P_0}{\partial z}, \tag{4.1}$$

$$\frac{\partial}{\partial r} rj_{r0} + \frac{\partial}{\partial \theta} j_{\theta 0} + r\frac{\partial}{\partial z} j_{z0} = 0, \tag{4.2}$$

$$\frac{\partial \alpha_0}{\partial t}+\boldsymbol{j}_0\cdot\nabla\alpha_0+s\frac{1}{r}\frac{\partial}{\partial r}[r^2\phi(\alpha_0)]=0, \tag{4.3}$$

$$-2(1+\epsilon\alpha_0)v_0=-\frac{\partial P_1}{\partial r}+sr\alpha_1, \tag{4.4}$$

$$2(1+\epsilon\alpha_0)u_0=-\frac{1}{r}\frac{\partial P_1}{\partial\theta}, \tag{4.5}$$

$$0=-\frac{\partial P_1}{\partial z}. \tag{4.6}$$

(For higher-order terms (2.6) should be used instead of (2.13). Additional terms should also be included in the relative-velocity formula (2.1).)

Equation (4.1), which expresses the hydrostatic balance of buoyancy and pressure, implies

$$\alpha_0=\alpha_0(r,t), \tag{4.7}$$

with $\alpha_0=a_0(t)$ as an important special case.

As expected, the azimuthal component of the pressure gradient balances the Coriolis term in (4.5). Moreover, the pressure is z-independent, and from (4.4), (4.6) and (4.7) a generalized 'thermal-wind' equation is obtained:

$$\frac{\partial}{\partial z}\{2[(1+\epsilon\alpha_0(r,t)]v_0+\epsilon sr\alpha_1\}=0. \tag{4.8}$$

Equations (4.7), (4.3) and (2.11) imply that, when α_0 genuinely depends on r,

$$u_0=u_0(r,t),\quad j_{r0}=j_{r0}(r,t). \tag{4.9}$$

In the special case $\alpha_0=\alpha_0(t)$ (4.3) decouples from the system because the second term is identically zero, and it follows that

$$u_0=u_0(r,\theta,t),\quad j_{r0}=j_{r0}(r,\theta,t). \tag{4.10}$$

Consequently the radial motion is always z-independent, but in this special situation there can be an azimuthal variation.

The azimuthal slip velocity of the dispersed particles is small and the *volume flux in the boundary layer on the barrier is presumed insignificant*. The appropriate boundary conditions are then those for an impermeable wall:

$$v_0=0\quad\text{at }\theta=0,2\pi. \tag{4.11}$$

These allow for some simplification if we again consider the azimuthally averaged flow variables, as defined by

$$\langle\psi\rangle=\frac{1}{2\pi}\int_0^{2\pi}\psi\,\mathrm{d}\theta. \tag{4.12}$$

For example, the average of (4.2) is

$$\frac{1}{r}\frac{\partial}{\partial r}r\langle j_{r0}\rangle+\frac{\partial}{\partial z}\langle j_{z0}\rangle=0. \tag{4.13}$$

Integration yields

$$\langle j_{z0}\rangle=-\left(\frac{1}{r}\frac{\partial}{\partial r}r\langle j_{r0}\rangle z\right)+C(r,t), \tag{4.14}$$

where the last term depends on the boundary conditions. Note that, when α_0 depends on r as well as on t, $\langle j_{r0}\rangle=j_{r0}$ in view of (4.9).

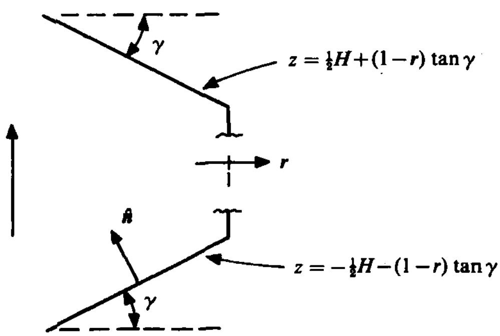

FIGURE 4. The geometry of conical inwardly inclined lids.

Appropriate boundary conditions on the cylindrical walls and inclined lids must now be given. For definiteness, let $\epsilon > 0$, so that the dispersed phase is heavier than the continuous phase and a sediment layer of thickness $S(\theta, z, t)$ forms at the outer cylindrical wall. *The volume flux within it will be assumed negligibly small*, which means that

$$\boldsymbol{j}\cdot\boldsymbol{n} = 0, \tag{4.15}$$

at $F(r, \theta, z, t) = r - 1 + S = 0$, with $\boldsymbol{n} = \nabla F$. The substitution of this equation in (2.10) and the fact that $\alpha_0 = \alpha_0(r, t)$ imply that $\boldsymbol{n}\cdot\boldsymbol{j}_{\mathrm{D}}$ is not a function of θ or z on the surface S. Therefore the thickness of the sediment layer on the cylindrical outer wall depends on time only: $S = S(t)$.

The sediment layer that forms on the inward inclined lid is also assumed to be thin. It follows that $\boldsymbol{j}\cdot\boldsymbol{n} = 0$ on these walls, where $\boldsymbol{n}$ is the appropriate normal vector. For the geometry shown in figure 4 this becomes

$$j_{r0} - j_{z0}\cot\gamma = 0 \quad \text{on } z = -\tfrac{1}{2}H - (1-r)\tan\gamma, \tag{4.16}$$

and

$$j_{r0} + j_{z0}\cot\gamma = 0 \quad \text{on } z = \tfrac{1}{2}H + (1-r)\tan\gamma. \tag{4.17}$$

The heavier phase is driven away from an outwardly inclined wall, and a particle-free layer appears there. An order-of-magnitude analysis shows that this layer is thin and inviscid if $E^{\frac{1}{2}}/\beta \ll 1$ and $|\epsilon| = O(1)$, and can be joined to the mixture core by an even thinner viscous sublayer. Since the mass flux in this sublayer is negligibly small, only clean fluid is transported from the core to the boundary layers, so that

$$\boldsymbol{j}_{\mathrm{D}}\cdot\boldsymbol{n} = 0$$

is the approximate condition at the wall. (A similar result was derived and used by Schneider (1982).) This can be written as

$$\boldsymbol{n}\cdot(\alpha\boldsymbol{j} + \phi(\alpha)\, r\hat{\boldsymbol{r}}) = 0; \tag{4.18}$$

for conical lids of inclination γ (figure 5)

$$\left(j_{r0} + \frac{\phi(\alpha_0)}{\alpha_0}\, r\right) + j_{z0}\cot\gamma = 0 \tag{4.19}$$

on $z = -\frac{1}{2}H + (1-r)\tan\gamma$, and

$$\left(j_{r0} + \frac{\phi(\alpha_0)}{\alpha_0}\, r\right) - j_{z0}\cot\gamma = 0 \tag{4.20}$$

on $z = \frac{1}{2}H - (1-r)\tan\gamma$.

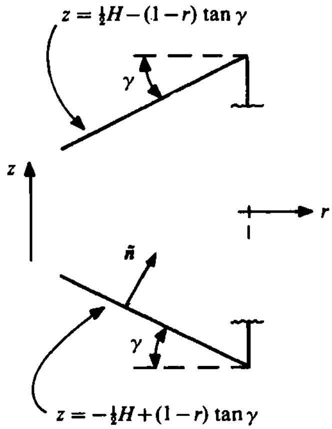

FIGURE 5. The geometry of conical outwardly inclined lids.

Since the inclination angle γ is independent of the azimuthal coordinate θ, the local values of j_{r_0} and j_{z_0} can be replaced by the averages $\langle j_{r_0}\rangle$ and $\langle j_{z_0}\rangle$ in (4.16), (4.17) and (4.19), (4.20).

The formulation of the basic problem is complete and the subscript 0 will be dropped from here on.

The substitution of (4.14) into the boundary conditions (4.16), (4.17) and (4.19), (4.20) yields two differential equations of the form

$$\left.\begin{aligned} &Q \pm \left[z(r)\left(\frac{\partial Q}{\partial r}\right)+rC(r,t)\right]\cot\gamma+\sigma r^2\frac{\phi(\alpha)}{\alpha} = 0,\\ \text{where}\qquad &Q = r\langle j_r\rangle; \end{aligned}\right\} \tag{4.21}$$

here $z(r)$ is the boundary surface of inclination γ, and σ is 0 and 1 for inward- and outward-inclined walls. These equations must be solved in conjunction with the reduced form of the volume-fraction equations

$$\frac{\partial \alpha}{\partial t}+\left[\frac{Q}{r}+(r\phi'(\alpha))\right]\frac{\partial \alpha}{\partial r}+2\phi(\alpha) = 0, \tag{4.22}$$

subject to the boundary conditions

$$Q = 0 \quad \text{at } r = 1-S(t),$$

where $\alpha = \alpha_{\mathrm{I}}$ at $t = 0$. (The initial conditions on j_r cannot be satisfied, because inertial terms, which are important for small t, have been neglected.)

Some illustrative results can be obtained when, to leading order, $\alpha = \alpha(t)$, which holds along the characteristics of (4.22). It follows that

$$t = -\frac{1}{2}\int_{\alpha_{\mathrm{I}}}^{\alpha(t)}\frac{\mathrm{d}\alpha}{\phi(\alpha)}. \tag{4.23}$$

The first configuration to be considered has symmetrically diverging inclined lids (figure 6). Mirror symmetry about the z-plane requires $j_z = 0$ at $z = 0$. The constant C in (4.14) is zero, and the substitution for $\langle j_z\rangle$ in (4.19) or (4.20) yields

$$\frac{\partial Q}{\partial r}+\frac{Q}{r+b} = -A(t)\frac{r^2}{r+b}, \tag{4.24}$$

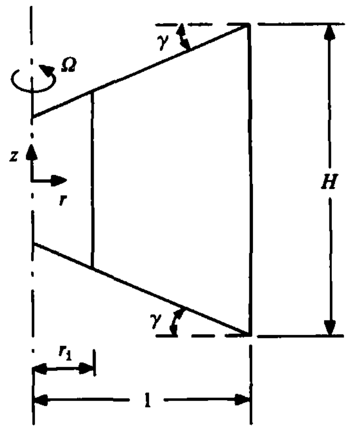

FIGURE 6. Cross-section of a z-symmetric container with outwardly inclined lids.

where
$$b = \tfrac{1}{2}H\cot\gamma - 1 \geqslant -r_1,$$

and
$$A(t) = \frac{\phi(\alpha(t))}{\alpha(t)}.$$

The solution of (4.24) that satisfies the boundary condition (4.15) is

$$Q = \tfrac{1}{3}A(t)\,\frac{[1-S(t)]^3 - r^3}{r+b}, \tag{4.25}$$

so that the averaged radial velocity of the dispersed phase is

$$\langle u_{\mathrm{D}}\rangle = A(t)\,r\left(1+\frac{1}{3}\,\frac{[1-S(t)]^3 - r^3}{r^2(r+b)}\right). \tag{4.26}$$

The first term in the parenthesis is just the basic settling velocity in a centrifugal force field, while the second arises from the flow caused by the inclined boundaries. For $b = O(1)$ both terms are comparable, since the width $S(t)$ of the sediment layer is small for the most part. In a long cylinder or for small inclination angles the geometrical parameter b is large, and the 'Boycott effect' diminishes accordingly. (The theory is probably invalid for large inclinations implied by $b \to -r_1$, especially in the proximity of r_1.)

The fact that $Q > 0$ at $r = r_1$ implies an injection into the centre of clean fluid returning on the outward inclined boundaries. This flux apparently occurs even for $r_1 \to 0$, which indicates that a pure-fluid core forms when no inner cylinder is present, provided b is small. (Again the validity of the present analysis is questionable for small r_1.)

The velocity U of the shock that is the surface of the outer sediment layer is determined from the continuity of volume flux:

$$\alpha(u_{\mathrm{D}} - U) = -U\alpha_{\mathrm{M}} \quad \text{at } r = 1 - S(t) \tag{4.27}$$

(where $u_{\mathrm{D}} = \langle u_{\mathrm{D}}\rangle$). But at this position $u_{\mathrm{D}} = A(t)\,r = r\phi(\alpha)/\alpha$, so that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -U = \frac{\phi(\alpha)\,r}{\alpha_{\mathrm{M}} - \alpha}, \tag{4.28}$$

with initial condition $S(0) = 0$. This equation can be integrated to obtain

$$S(t) = 1-\left[\frac{\alpha_{\mathrm{M}}-\alpha_{\mathrm{I}}}{\alpha_{\mathrm{M}}-\alpha(t)}\right]^{\frac{1}{2}}. \tag{4.29}$$

The velocity of the shock that separates the mixture from the region of purified fluid is simply the velocity u_{D} of the particles that were on the inner solid wall $r = r_{\mathrm{i}}$ at $t = 0$. Consequently the azimuthally averaged position of this shock is defined by

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \langle u_{\mathrm{D}}\rangle\,|_{r=R}, \tag{4.30}$$

with $R = r_{\mathrm{i}}$ at $t = 0$. Integration of the system of equations (4.23), (4.26), (4.29) and (4.30) produces the main features of this time-dependent separation process.

A rough comparison can be made of the separative efficiency of two sectioned containers of the type shown in figure 6. Containers A and B have identical volumes V, as well as inner and outer radii, with lids that are uninclined ($\gamma = 0$) and inclined ($\gamma \neq 0$) respectively. Obviously the heights of containers A and B are properly adjusted, with $H_{\mathrm{B}} > H_{\mathrm{A}}$.

The volume of separated heavier phase at time t is $2\pi H\alpha_{\mathrm{M}} S(t)$ and the initial total bulk of this constituent in the mixture is $\alpha_{\mathrm{I}} V$. The separated fraction $F(t)$ is, assuming a thin sediment layer,

$$F(t) = \frac{\alpha_{\mathrm{M}}}{\alpha_{\mathrm{I}}}\,2\pi H\,\frac{S(t)}{V}. \tag{4.31}$$

In view of (4.29), the thickness of the sediment layer $S(t)$ is the same in both containers, so that

$$\frac{F_{\mathrm{B}}(t)}{F_{\mathrm{A}}(t)} = \frac{H_{\mathrm{B}}}{H_{\mathrm{A}}}, \tag{4.32}$$

where the subscripts refer to the specific containers. The improvement ratio is a constant larger than 1, which has a maximum value when $\gamma = \tan^{-1}[H/2(1-r_{\mathrm{i}})]$ of $3(1+r_{\mathrm{i}})/(2+r_{\mathrm{i}}) < 2$. If r_{i} is small then $F_{\mathrm{B}}/F_{\mathrm{A}} < 1.5$.

The second configuration to be discussed is a container of constant height between parallel inclined lids of inclination angle γ (figure 7).

The substitution of (4.14) into (4.16) and (4.20) yields

$$-\left[\left(\frac{1}{r}\frac{\partial}{\partial r}Q\right)(\tfrac{1}{2}H+(1-r)\tan\gamma)+C(r,t)\right]\cot\gamma+\frac{Q}{r} = 0, \tag{4.33}$$

$$-\left[\left(\frac{1}{r}\frac{\partial}{\partial r}Q\right)(-\tfrac{1}{2}H+(1-r)\tan\gamma)+C(r,t)\right]\cot\gamma+\frac{Q}{r}+\frac{\phi(\alpha)}{\alpha}r = 0. \tag{4.34}$$

The result of subtracting (4.33) from (4.34) is

$$\left(\frac{1}{r}\frac{\partial}{\partial r}Q\right)H\cot\gamma+\frac{\phi(\alpha)}{\alpha}r = 0. \tag{4.35}$$

The solution accounting for the boundary condition (4.15) is

$$Q = \frac{A(t)}{3H\cot\gamma}([1-S(t)]^3-r^3), \tag{4.36}$$

where $A(t) = \phi(\alpha)/\alpha$. The averaged radial velocity of the dispersed phase is therefore

$$\langle u_{\mathrm{D}}\rangle = A(t)\,r\left[1+\frac{[1-S(t)]^3-r^3}{3Hr^2\cot\gamma}\right]. \tag{4.37}$$

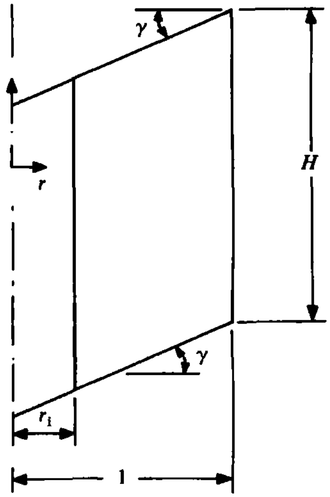

FIGURE 7. Cross-section of a container of constant height with inclined lids.

$\frac{1}{2}H\cot\gamma$	∞	1.0	0.5	0.2	0.1
F_B/F_A	1	1.16	1.28	1.61	2.11
$\bar{t}$	1.50	0.87	0.68	0.44	0.30

TABLE 1

As in the previous case, the last term in (4.37) is the enhanced settling, due to the inclined boundaries. Its contribution is significant for short containers with steeply inclined lids, and is negligibly small for long containers or small inclination angles. Additional details of the separation process can be calculated as indicated for the previous case.

Separative performance can again be compared for sectioned containers A and B of identical volume, the same inner and outer radii but with uninclined and inclined lids. (Note that here casings A and B now have the same height.) The volume of the dispersed phase in the mixture, which was initially $\alpha_I \pi H(1-r_i^2)$, is at time t reduced to $\alpha(t)\pi(1-R^2(t))$, where $R(t)$ is the position of the shock between the mixture and the clean fluid given by (4.30). The fraction of the dispersed phase that has separated out from the mixture at any time is then

$$F(t) = 1-\frac{\alpha(t)}{\alpha_I}\frac{1-R^2(t)}{1-r_i^2}, \qquad (4.38)$$

The ratio $\alpha(t)/\alpha_I$ is the same for both containers, and in the dilute limit $\alpha_I \rightarrow 0$, which is considered here for simplicity, its value is $\exp(-2t)$.

The improvement ratio F_B/F_A is a function of t in this case. To illustrate the effect of the inclined lids this ratio is evaluated at time $\bar{t}$, when $F_B(\bar{t}) = 0.95$, for containers with $r_i = 0$. Typical values are given in table 1.

5. Conclusions

The time-dependent separation process in a large rotating cylindrical container with inclined lids has been considered under the assumptions that inertial forces are negligible and viscous shear is important only in boundary layers. In axisymmetric containers, the Coriolis acceleration dominates the motion in the core, where there can be no radial mass flux. In this case, the geometry of the endcaps cannot much affect the basic sedimentation velocity of any interior particle. Moreover, the pure-fluid layer expands as the particles on the interface settle. The wall area available for particle collection also decreases significantly with time which essentially eliminates the basis of the Boycott effect. The strong buoyancy force in the growing pure-fluid layer is balanced by the centrifugal pressure gradient that arises from the production of vortical motions in different regions of the mixture.

In gravitational settling, the pure fluid is confined to a thin boundary layer where buoyancy can be balanced by shear forces. Very much the same is true for settling in a sectioned cylinder in which there is insignificant relative rotation of the fluid because the azimuthal motion is physically blocked. The Coriolis force is then unimportant and dynamical equilibrium once again requires strong shear in a thin jet to counteract local buoyancy.

The motion in the sectioned cylinder is controlled by the boundary layers on the lids. When the mass flux of the dispersed component is negligibly small in these layers, the inclination of the lids can enhance the separation in a manner similar to that in a gravitational field. The azimuthal pressure gradient in the core sustains a large z-independent radial velocity; the volume fraction depends on r and t, and the thickness of the sediment layer on the sidewall is a function of t only.

The Boycott effect in gravitational sedimentation is usually, and correctly, attributed to an increase of settling area and a shortening of settling distances. The pure-fluid layer that forms is in this view secondary. For centrifugal settling, however, the reverse is the more accurate generalization: unless a pure-fluid layer forms in a container, little or no geometrical enhancement of settling can occur.

In general, centrifugal settling in a container is essentially unaffected by shape whenever the Coriolis force is dominant. Geometrical enhancement of settling depends then on the extent to which this force can be counteracted. Practical methods to do this would make use of large shear forces between closely spaced plates (Bark & Johansson 1982) or with meridional sections as described here.

Although the present work deals with batch rather than continuous processing, the main conclusions have rather obvious application to centrifuge design. This will be discussed separately.

Appendix

Some useful kinematic relations between velocities and volume fluxes are summarized below:

$$\boldsymbol{q}_{\mathrm{R}} = \boldsymbol{q}_{\mathrm{D}} - \boldsymbol{q}_{\mathrm{C}}, \tag{A 1}$$

$$(1+\epsilon\alpha)\,\boldsymbol{q} = (1-\alpha)\,\boldsymbol{q}_{\mathrm{C}} + \alpha(1+\epsilon)\,\boldsymbol{q}_{\mathrm{D}}, \tag{A 2}$$

$$\boldsymbol{q}_{\mathrm{D}} = \boldsymbol{q} + \frac{1-\alpha}{1+\epsilon\alpha}\,\boldsymbol{q}_{\mathrm{R}}, \tag{A 3}$$

$$\boldsymbol{q}_{\mathrm{C}} = \boldsymbol{q} - \frac{\alpha(1+\epsilon)}{1+\epsilon\alpha}\,\boldsymbol{q}_{\mathrm{R}}, \tag{A 4}$$

$$j = \alpha q_{\mathrm{D}} + (1-\alpha)\, q_{\mathrm{C}}, \tag{A 5}$$

$$j = q - \epsilon \frac{\alpha(1-\alpha)}{1+\epsilon\alpha}\, q_{\mathrm{R}}, \tag{A 6}$$

$$j_{\mathrm{D}} = \alpha j + \alpha(1-\alpha)\, q_{\mathrm{R}}, \tag{A 7}$$

$$j_{\mathrm{D}} = \alpha q + \frac{\alpha(1-\alpha)}{1+\epsilon\alpha}\, q_{\mathrm{R}}, \tag{A 8}$$

$$j_{\mathrm{D}} = \alpha q_{\mathrm{D}}. \tag{A 9}$$

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